The heat equation

The heat equation with *zero ends* boundary conditions models the temperature of an (insulated) wire of length L:

$$\left\{\begin{array}{c} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u(0,t) = u(L,t) = 0. \end{array}\right.$$

Here $u(x,t)$ denotes the temperature at a point x on the wire at time t. The initial temperature $f(x)$ is specified by the equation

$$u(x,0) = f(x).$$

Method:

- Find the sine series of $f(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L})) \exp(-k(\frac{n\pi}{L})^2 t).$$

Example: Let

$$f(x) = \left\{\begin{array}{ll} -1, & 0 \le t \le \pi/2, \\ 2, & \pi/2 < t < \pi. \end{array}\right.$$

Then $L = \pi$ and

$$b_n(f) = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx = -2\,\frac{2\,\cos(n\pi) - 3\,\cos(1/2\,n\pi) + 1}{n}.$$

Thus

$$f(x) \sim b_1(f)\sin(x) + b_2(f)\sin(2x) + ... = \frac{2}{\pi}\sin(x) - \frac{6}{\pi}\sin(2x) + \frac{2}{3\pi}\sin(3x) + ...$$

The function $f(x)$, and some of the partial sums of its sine series, looks like Figure 1.

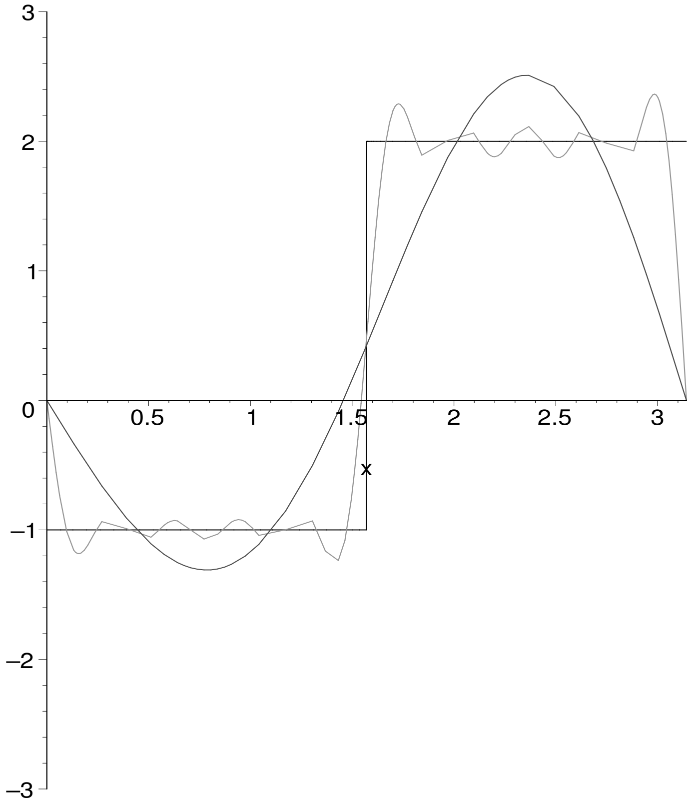

Figure 1: $f(x)$ and two sine series approximations.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the heat equation, therefore, is

$$u(x,t) = \sum_{n=1}^{\infty} (b_n(f) \sin(\frac{n\pi x}{L})) \exp(-k(\frac{n\pi}{L})^2 t).$$

Taking the first 60 terms of this series, the graph of the solution at $t=0$, $t=0.5$, $t=1$, looks approximately like Figure 2.

The heat equation with *insulated ends* boundary conditions models the temperature of an (insulated) wire of length L:

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u_x(0,t) = u_x(L,t) = 0. \end{cases}$$

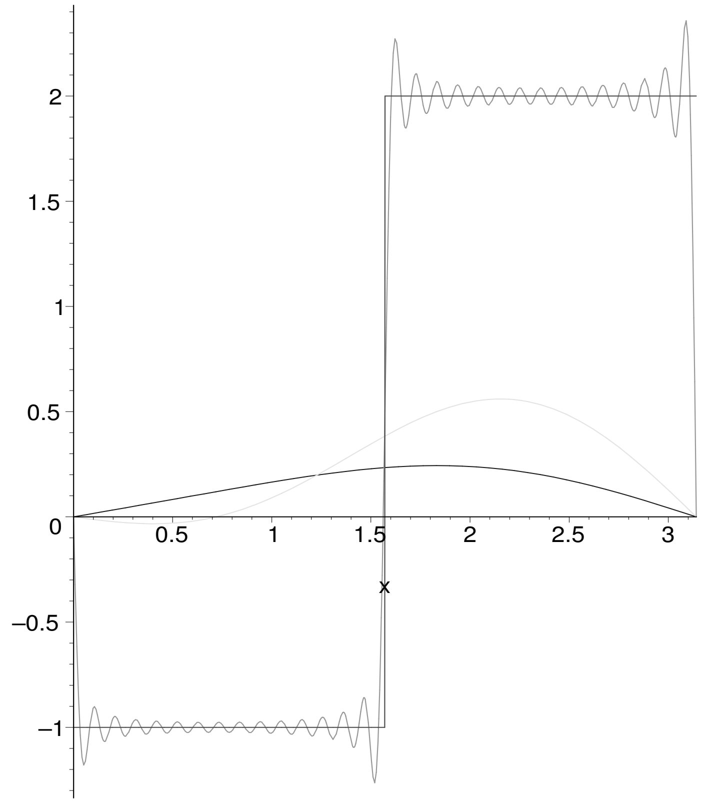

Figure 2: $f(x)$, $u(x,0)$, $u(x,0.5)$, $u(x,1.0)$ using 60 terms of the sine series.

Here $u_x(x,t)$ denotes the partial derivative of the temperature at a point x on the wire at time t. The initial temperature $f(x)$ is specified by the equation $u(x,0) = f(x)$.

Method:

- Find the cosine series of $f(x)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f) \cos(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f) \cos(\frac{n\pi x}{L})) \exp(-k(\frac{n\pi}{L})^2 t).$$

Example: Let

$$f(x) = \begin{cases} -1, & 0 \le t \le \pi/2, \\ 2, & \pi/2 < t < \pi. \end{cases}$$

Then $L = \pi$ and

$$a_n(f) = \frac{2}{\pi} \int_0^{\pi} f(x)\cos(nx)dx = -6\frac{\sin\left(\frac{1}{2}\pi n\right)}{\pi n},$$

for $n > 0$ and $a_0 = 1$. Thus

$$f(x) \sim \frac{a_0}{2} + a_1(f)\cos(x) + a_2(f)\cos(2x) + ...$$

The function $f(x)$, and some of the partial sums of its cosine series, looks like Figure 3.

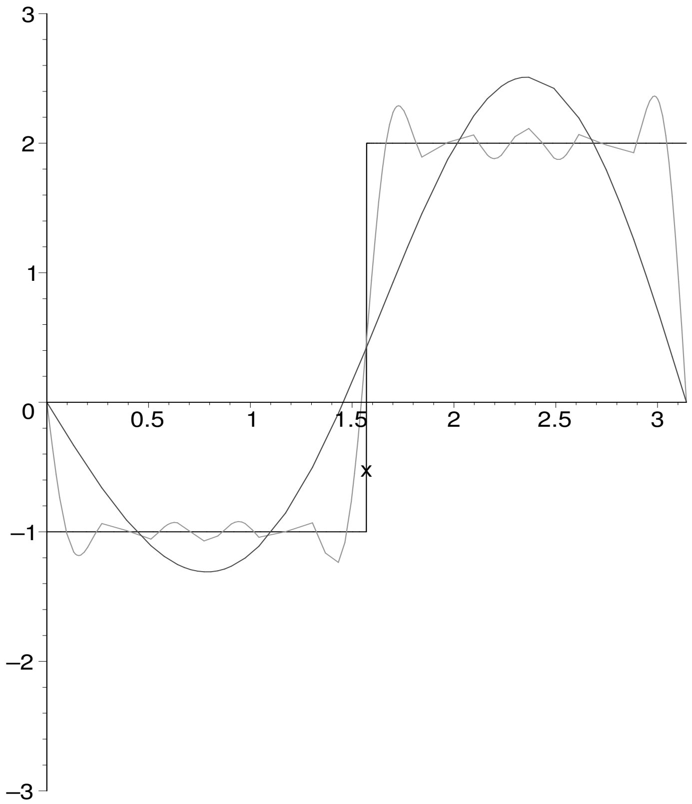

Figure 3: $f(x)$ and two cosine series approximations.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the heat equation, therefore, is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n(f)\cos(\frac{n\pi x}{L}))\exp(-k(\frac{n\pi}{L})^2 t).$$

Taking only the first 30 terms of this series, the graph of the solution at $t = 0$, $t = 0.5$, $t = 1$, looks approximately like:

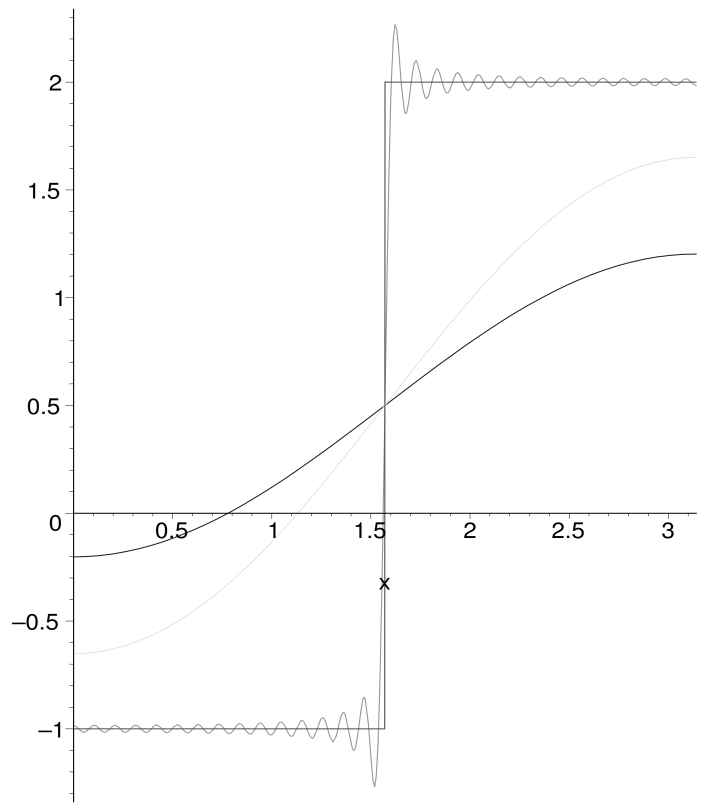

Figure 4: $f(x)$, $u(x,0)$, $u(x,0.5)$, $u(x,1.0)$ using 60 terms of the cosine series.